TOPOLOGICAL COMPLEXITIES OF SURFACES

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ABSTRACT. The sphere S^2 and the torus T^2 are the only closed connected surfaces for which higher topological complexities are known (for each $n \in \{2, 3, ...\} \subset \mathbb{N}$, $\mathrm{TC}_n(S^2) = n$ and $\mathrm{TC}_n(T^2) = 2n - 2$). This text aims to find topological complexities for most other closed connected surfaces. For all but S^2, T^2, the projective plane ($\mathbb{RP}^2$) and the Klein bottle the n-th topological complexity is $2n$.

1. INTRODUCTION

In robot motion planning there is a very natural question: What is the minimum number of instructions to determine a path between any two positions in the space of configurations of a robot? Michael Farber defined the topological complexity of a space (TC) in the early 2000s, in order to make this problem concrete. And he was able to find a homotopy invariant solution (TC) [Fa03]. In 2010, Yuli Rudyak defined a generalization of this concept (TC_n) [Ru10]. This new sequence of homotopy invariants allowed for any number of two or more positions in the configuration space. Their definition can be formulated through the genus of Schwarz [Sv66], and in it n represents the number of ordered positions the robot needs to pass through in its motion inside the configuration space X.

1.1. **Definition.** For each $n \in \{2, 3, ...\} \subset \mathbb{N}$ the n-th topological complexity of a space $\mathrm{TC}_n(X)$ is the normalized Schwarz genus of a fibrational substitute of the diagonal map $\Delta_n : X \to X^n$.

Since TC was first defined using the non-normalized Schwarz genus, in this definition TC_2 is known as the normalized topological complexity ($\mathrm{TC} - 1$)
In the entire text, the word "surfaces" refers to closed connected surfaces in particular, that is, compact connected 2-manifolds without boundary. The following two sections present the higher topological complexities of all orientable surfaces of genus g greater than 1 ($\mathrm{TC}_n(M_g)$) (Section 2), and the normalized topological complexity of non-orientable surfaces of genus greater than 2 ($\mathrm{TC}_2(N_g)$) (Section 3). Finally, Section 4 generalizes the result in Section 3 by giving

all higher topological complexities of non-orientable surfaces of genus greater than 2.
The problem just mentioned on the normalized topological complexity of non-orientable surfaces was suggested by M. Farber and M. Grant in the problem section at the *Arbeitsgemeinschaft: Topological Robotics* [OR].

2. Higher Topological Complexities of Orientable Surfaces

The notation M_g stands for any orientable surface of genus $g \geq 2$. Theorem 2.1 below shows the topological complexities of such surfaces. This is a generalization of Michael Farber's result $\mathrm{TC}_2(M_g) = 4$ [Fa03, Theorem 3].

2.1. **Theorem.** *For any $g \geq 2$,*

$$\mathrm{TC}_n(M_g) = 2n.$$

Proof. The covering dimension helps bound our target from above, $\mathrm{TC}_n(M_g) \leq n \dim(M_g) = 2n$. On the other hand, the lower bound $2n \leq \mathrm{TC}_n(M_g)$ is achieved by using the length of the largest nontrivial cup product. There are cohomology classes u, v, x and y in $H^1(M_g; \mathbb{F})$ such that $u^2 = 0$, $v^2 = 0$, $x^2 = 0$, $y^2 = 0$ and also $u \smile v = x \smile y = a \neq 0$, where $a \in H^2(M_g; \mathbb{F})$ is the fundamental class (Figure 1). Taking the projection onto the i-th factor, $p_i : (M_g)^n \to M_g$, there are $u_i = p_i^* u$, $v_i = p_i^* v$, $x_i = p_i^* x$ and $y_i = p_i^* y$ for each $i \in \{1, \ldots, n\}$ such that we can write the following product

$$(u_1 - u_2) \smile \cdots \smile (u_1 - u_n)(v_1 - v_2) \smile \cdots \smile$$

$$(v_1 - v_n) \smile (x_1 - x_2) \smile (y_1 - y_2)$$

Allowing the projections $u_1 = 0$, $v_1 = 0$, $x_2 = 0$ and $y_2 = 0$ the product is

$$u_2 \smile \cdots \smile u_n v_2 \smile \cdots \smile v_n \smile x_1 \smile y_1 =$$

$$u_2 \smile v_2 \smile \cdots \smile u_n \smile v_n \smile x_1 \smile y_1 \neq 0$$

and hence by [Ru10, Proposition 3.4] $\mathrm{TC}_n(M_g) \geq 2n$. $\square$

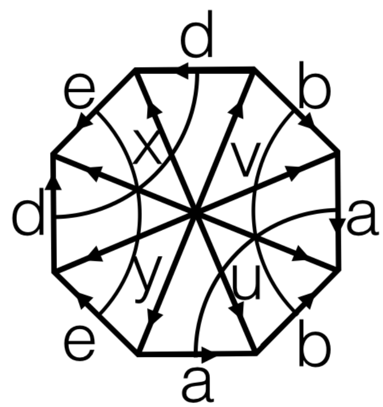

FIGURE 1. Orientable surface of genus 2.

3. Topological Complexity of Nonorientable Surfaces

As mentioned in the previous section, Michael Farber originally showed that the normalized topological complexity of any orientable surface is 4 in [Fa03, Theorem 9]. It is also true that the normalized topological complexity of non-orientable surfaces is 4. In 2010 in Oberwolfach a gathering of mathematicians working on the field of topological robotics asked this question. In particular, Michael Farber and Mark Grant proposed it in the problem section at the *Arbeitsgemeinschaft: Topological Robotics* published in [OR]. Let N_g represent any non-orientable surface of genus $g \geq 3$. At the time of the conference the inequality $3 \leq \mathrm{TC}_2(N_g) \leq 4$ was known. As in the orientable case, the upper bound is given by the covering dimension. The lower bound however is improved to 4 in the following theorem (Theorem 3.1, it is probably known in the algebraic topology folklore but has never been written down until now).

3.1. **Theorem.** *For any $g \geq 3$,*

$$\mathrm{TC}_2(N_g) = 4.$$

Proof. Again, the lower bound will be settled by finding a non-zero cup product of length 4. The non-orientable surface of genus 3 can be realized as follows $N_3 = T^2 \# \mathbb{RP}^2$ (Figure 2). Then, the choice of cohomology classes can be given as $u, v, w \in H^2(N; \mathbb{Z}/2\mathbb{Z})$ with u and v representing the T^2 part and w representing the $\mathbb{RP}^2$ part. Now, for the projections, $p_i : (N_g)^2 \to N_g$ with $i \in \{1, 2\}$, there are $u_i = p_i^* u$,

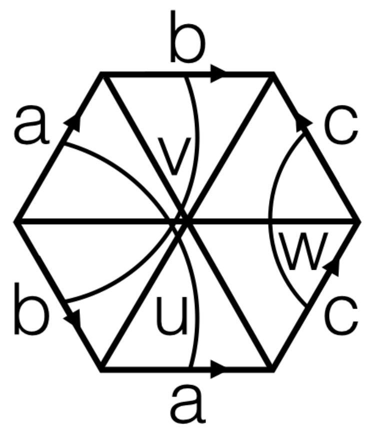

FIGURE 2. Non-orientable surface of genus 3.

$v_i = p_i^* v$ and $w_i = p_i^* w$ such that there is the nonzero product

$$(u_1 - u_2) \smile (v_1 - v_2) \smile (w_1^2 - w_2^2).$$

This product is non-zero since the following projections, $u_1 = 0$, $v_1 = 0$ and $w_2 = 0$, turn the product into

$$u_2 \smile v_2 \smile c_1^2 \neq 0$$

□

4. Higher Topological Complexities of Nonorientable Surfaces

This section is dedicated to show a generalization of the result in Section 3, namely $\mathrm{TC}_n(N_g) = 2n$ for all $n \geq 2$. The proof of Theorem 4.1 follows from the proof Theorem 3.1.

4.1. **Theorem.** *For any $g \geq 3$,*

$$\mathrm{TC}_n(N_g) = 2n.$$

Proof. Similarly to the previous theorem, Theorem 3.1, there are cohomology classes $u, v, w \in H^2(N; \mathbb{Z}/2\mathbb{Z})$ for which using the induced maps of the natural projections, as done before, there is the nonzero product

$$(u_1 - u_2) \smile \cdots \smile (u_1 - u_n) \smile (v_1 - v_2) \smile \cdots \smile (v_1 - v_n) \smile (w_1^2 - w_2^2).$$

Again using the same projections, $u_1 = 0$, $v_1 = 0$ and $w_2 = 0$, the product turns into

$$u_2 \smile \cdots \smile u_n \smile v_2 \smile \cdots \smile v_n \smile w_1^2 =$$
$$u_2 \smile v_2 \smile \cdots \smile u_n \smile v_n \smile w_1^2 \neq 0$$

$\square$

The previous theorems settle the topological complexities for almost every surface. One of the questions of this kind for which there is still no answer is: What is the normalized topological complexity of the Klein bottle? And also: What are the higher topological complexities of $\mathbb{RP}^2$ and the Klein bottle?

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